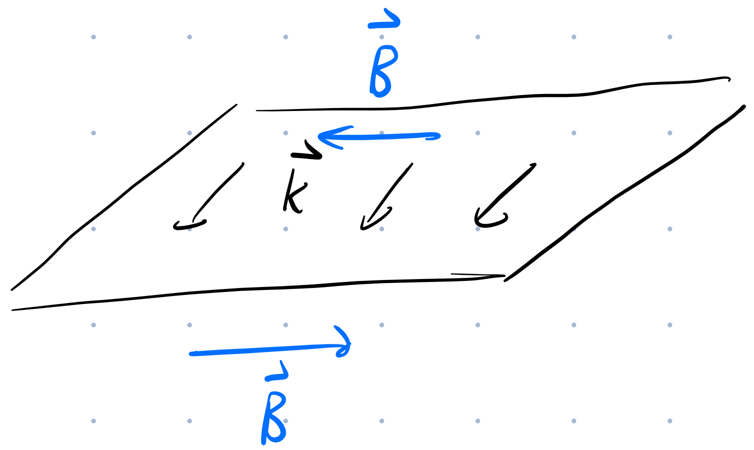
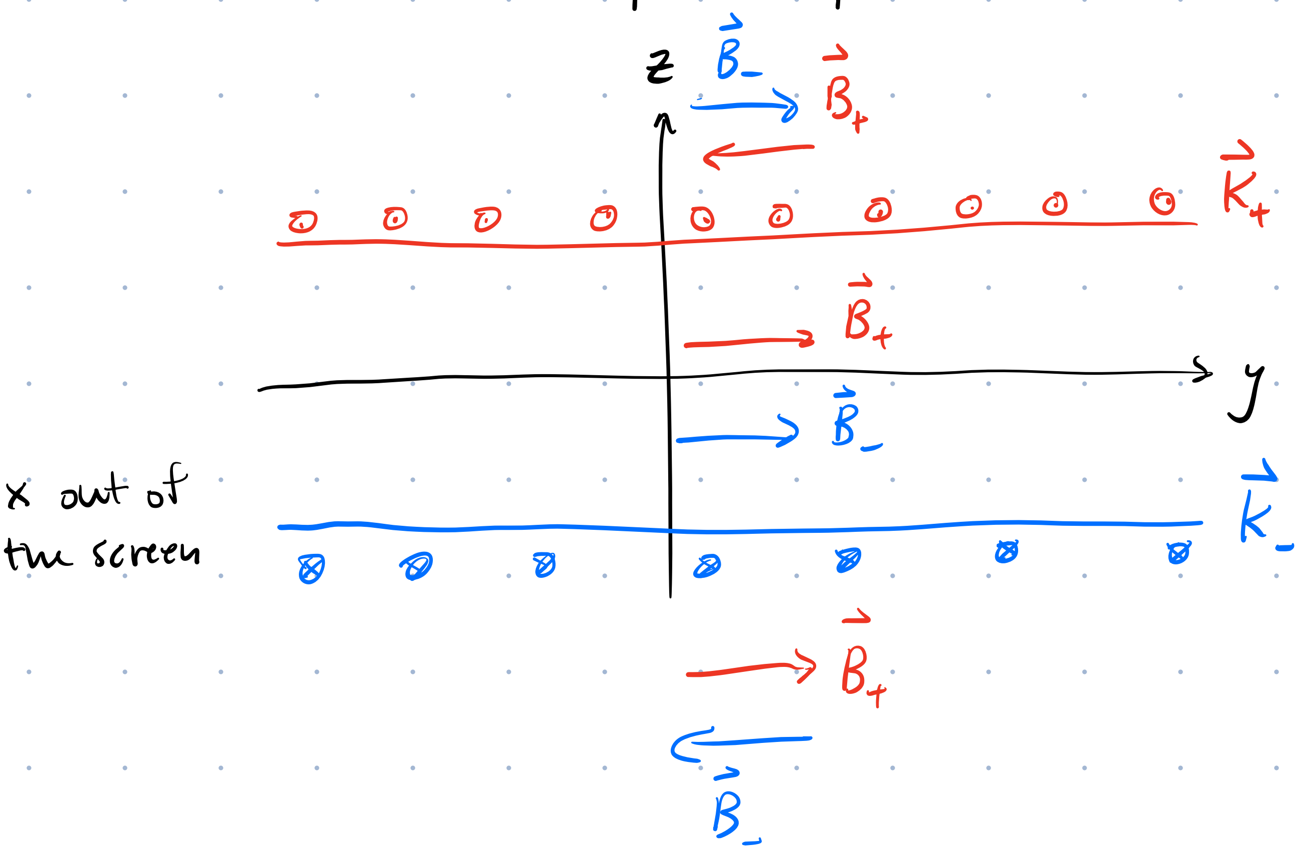
PHYS 301 Tutorial #8 Group Prob. Sol'ns Nov. 15, 2024

1.



$B = \frac{\mu_0 K}{2}$

(a) Side View of a pair of parallel sheets.



Between the sheets, the $\vec{B}_+$ & $\vec{B}_-$ contributions add constructively s.t.

$$\boxed{\vec{B}_{in} = \mu_0 K \hat{y}} \quad (\text{for } -a < z < a)$$

Above and below, the two contribution cancel s.t. $\boxed{\vec{B}_{out} = 0}$ (for $z > a$ and $z < -a$)

(b) Know that $\vec{B} = \vec{\nabla} \times \vec{A}$.

$\therefore$ Given $\vec{B}$, we can try to deduce $\vec{A}$,

For infinite sheets in x- & y-directions, expect $\vec{A} = \vec{A}(z)$.

Furthermore, we usually expect $\vec{A}$ to be parallel or antiparallel to the current.

$$\therefore \vec{A} = A(z)\,\hat{x}$$

$$\therefore \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A(z) & 0 & 0 \end{vmatrix} = \frac{dA}{dz}\,\hat{y}$$

So we have $$\frac{dA}{dz} = \begin{cases} 0 & z < -a \\ \mu_0 K & -a < z < a \\ 0 & z > a \end{cases}$$

(i) Start w/ $-a < z < a$

$$dA = \mu_0 k\, dz \rightarrow \boxed{\vec{A} = \mu_0 k z\, \hat{x}}$$

Note that z changes sign when z passes through zero.

Clearly $\vec{\nabla} \times \vec{A} = \mu_0 k\, \hat{y} = \vec{B}$ as required. ✓

It is also clear that $\vec{\nabla} \cdot \vec{A} = 0$ ✓

(ii) Next, consider $z > a$ s.t. $B = 0$.

In this case $\vec{\nabla} \times \vec{A} = 0$ and we require $\vec{A}$ to be continuous at $z = +a$

From (i) $\vec{A} = \mu_0 k a\, \hat{x}$ at $z = a$.

$\therefore$ we take $\boxed{\vec{A} = \mu_0 k a\, \hat{x}}$ $\forall\ z \geq a$.

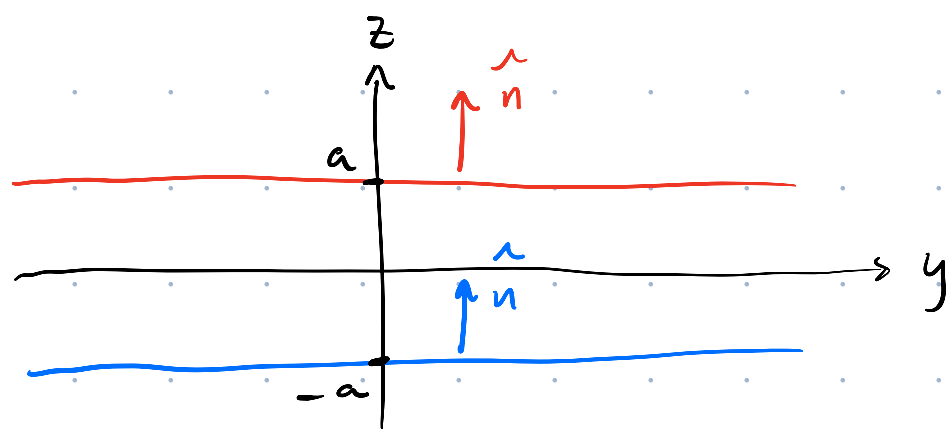
Likewise, by the same reasoning,

$$\boxed{\vec{A} = -\mu_0 k a\, \hat{x}}\ \forall\ z \leq -a$$

(c) the other boundary condition on $\vec{A}$ is

$$\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K}$$

where n is in the dir'n $\perp$ to $\vec{K}$. Take our normal vectors to point along $+\hat{z}$ dir'n.



Start w/ $z = +a$. $\quad n = z$

$$\therefore \frac{\partial \vec{A}_{above}}{\partial z} - \frac{\partial \vec{A}_{below}}{\partial z} = -\mu_0 K \hat{x}$$

$$\Rightarrow \cancelto{0}{\frac{\partial}{\partial z}\left(\mu_0 K a \hat{x}\right)} - \underbrace{\frac{\partial}{\partial z}\left(\mu_0 K z \hat{x}\right)}_{-\mu_0 K \hat{x}} = -\mu_0 K \hat{x}$$

Next, consider $z = -a$ $\quad\quad n = z$

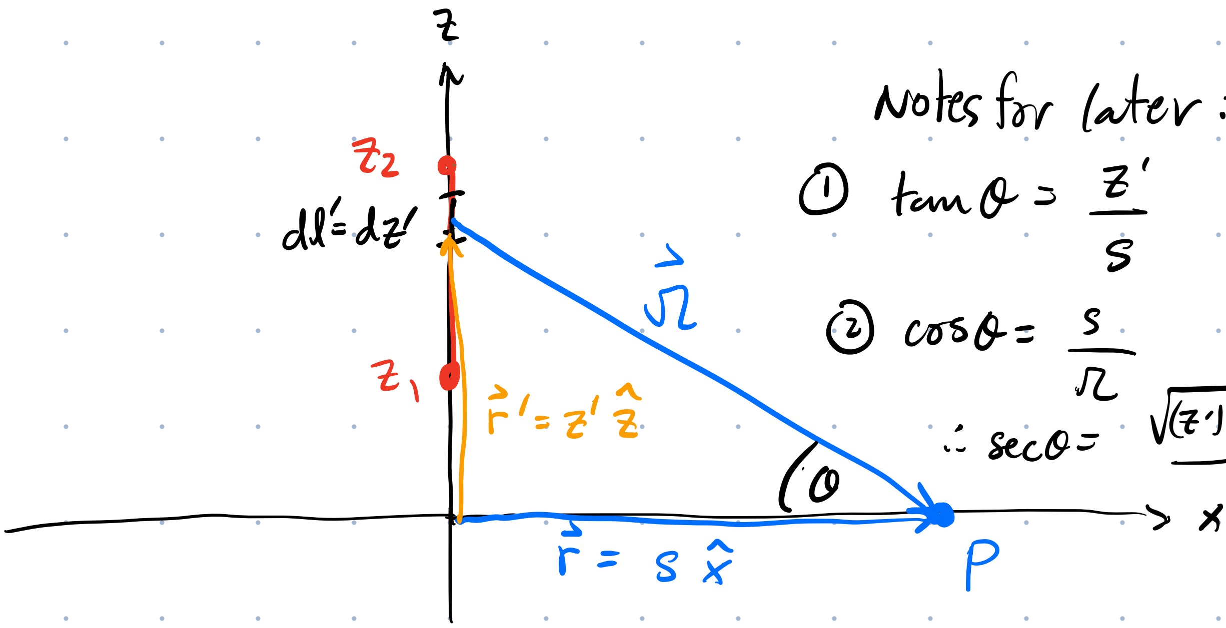
$$\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = +\mu_0 K \hat{x}$$

b/c $\vec{K}$ is in the $-\hat{x}$ dir'n.

$$\underbrace{\frac{\partial}{\partial z}\left(\mu_0 K z \hat{x}\right)}_{\mu_0 K \hat{x}} - \cancelto{0}{\frac{\partial}{\partial z}\left(-\mu_0 K a \hat{x}\right)} = \mu_0 K \hat{x} \quad ✓$$

Note that "above" means the region of space that $\hat{n}$ points towards.

2.



Notes for later:

① $\tan\theta = \dfrac{z'}{s}$

② $\cos\theta = \dfrac{s}{\mathscr{R}}$

$\therefore \sec\theta = \dfrac{\sqrt{(z')^2 + s^2}}{s}$

We can arbitrarily place our "field pt" P in on the x-axis

Use cylindrical coords.

From the diagram $\vec{r} = s\,\hat{s}$

$\vec{r}\,' = z'\,\hat{z}$

$\therefore \vec{\mathscr{R}} = \vec{r} - \vec{r}\,' = s\,\hat{s} - z'\,\hat{z}$ $\qquad\qquad \vec{I} = I\,\hat{z}$

$\mathscr{R} = \sqrt{(z')^2 + s^2}$

To calc. $\vec{A}$, we use $\quad \vec{A} = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\vec{I}\,dl'}{\mathscr{R}}$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{z_1}^{z_2} \frac{I\,dz'\;\hat{z}}{\sqrt{(z')^2 + s^2}}$$

make the substitution $z' = s\tan\theta$

$$\therefore \quad dz' = s\sec^2\theta \, d\theta$$

$$\sqrt{(z')^2 + s^2} = s\sqrt{\tan^2\theta + 1} = s\sec\theta$$

$$\therefore \vec{A} = \frac{\mu_0 I \hat{z}}{4\pi}\int_{\theta_1}^{\theta_2} \frac{\cancel{s}\sec^{\cancel{2}}\theta \, d\theta}{\cancel{s}\cancel{\sec\theta}} = \frac{\mu_0 I \hat{z}}{4\pi}\underbrace{\int_{\theta_1}^{\theta_2} \sec\theta \, d\theta}_{\ln|\sec\theta + \tan\theta|}$$

(from integral table)

$$\therefore \vec{A} = \frac{\mu_0 I \hat{z}}{4\pi}\ln\left|\underbrace{\frac{\sqrt{(z')^2+s^2}}{s}}_{\text{positive}} + \underbrace{\frac{z'}{s}}_{\text{positive}}\right|\Bigg|_{z=z_1}^{z_2}$$

can drop abs. value sign

$$\therefore \vec{A} = \frac{\mu_0 I}{4\pi}\left\{\ln\left(\frac{\sqrt{z_2^2+s^2}}{s} + \frac{z_2}{s}\right) - \ln\left(\frac{\sqrt{z_1^2+s^2}}{s} + \frac{z_1}{s}\right)\right\}\hat{z}$$

$$\boxed{\therefore\ \vec{A} = \frac{\mu_0 I}{4\pi} \ln\left(\frac{\sqrt{z_2^2+s^2}+z_2}{\sqrt{z_1^2+s^2}+z_1}\right)\hat{z}}$$

To find $\vec{B}$, we can take $\vec{\nabla}\times\vec{A}$ in cylindrical coords.

3. $$\vec{A}_{in} = \frac{\mu_0 R \omega \sigma}{3} r \sin\theta \, \hat{\phi}$$

$$\vec{A}_{out} = \frac{\mu_0 R^4 \omega \sigma}{3} \frac{\sin\theta}{r^2} \hat{\phi}$$

(a) $$\vec{B}_{in} = \vec{\nabla} \times \vec{A}_{in} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta \, A_\phi\right)\right]\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(r A_\phi\right)\hat{\theta}$$

$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\mu_0 R\omega\sigma}{3} r\sin^2\theta\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\mu_0 R\omega\sigma}{3} r^2 \sin\theta\right)\hat{\theta}$$

$$= \frac{1}{\cancel{r}\cancel{\sin\theta}} \frac{\mu_0 R\omega\sigma}{3} 2\cancel{r}\cancel{\sin\theta}\cos\theta \, \hat{r}$$

$$- \frac{1}{\cancel{r}} \frac{\mu_0 R\omega\sigma}{3} 2\cancel{r}\sin\theta \, \hat{\theta}$$

$$= \frac{2\mu_0 R\omega\sigma}{3}\left(\underbrace{\cos\theta \, \hat{r} - \sin\theta \, \hat{\theta}}_{\hat{z}}\right)$$

$$\boxed{\therefore\ \vec{B}_{in} = \frac{2}{3}\mu_0 R\omega\sigma\ \hat{z}}$$

In Assign. #4, you will find $\vec{B}$ at the centre of a rotating spherical shell using the Biot-Savart law. You should find this result.

(b) If the solid sphere has charge density $\rho = \frac{Q}{\frac{4}{3}\pi R^3}$,

then the charge on a spherical shell of radius r' & thickness dr' is:

$$Q_{shell} = \rho\, 4\pi r'^2 dr'$$

$$= \frac{Q}{\frac{4}{3}\pi R^3} 4\pi r'^2 dr'$$

s.t. $\sigma = \frac{Q_{shell}}{4\pi r'^2} = \frac{Q\,dr'}{\frac{4}{3}\pi R^3} = \boxed{\rho\, dr'}$

$$\therefore\ d\vec{A}_{in} = \frac{\mu_0 r' \omega \rho\, dr'}{3}\, r \sin\theta\, \hat{\phi}$$

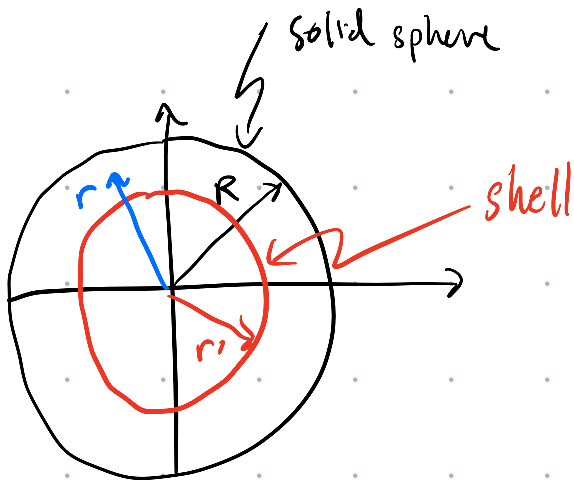
$$d\vec{A}_{out} = \frac{\mu_0 r'^4 \omega \rho\, dr'}{3}\, \frac{\sin\theta}{r^2}\, \hat{\phi}$$

(c) Add up the shells to construct a sphere of radius R.

$$\vec{A}_{sphere} = \int_0^r d\vec{A}_{in} + \int_r^R d\vec{A}_{out}$$

$$= \frac{\mu_0 \omega \rho}{3} \sin\theta\, \hat{\phi} \left[\overbrace{\int_r^R r' r\, dr'}^{r' > r} + \overbrace{\int_0^r \frac{r'^4}{r^2}\, dr'}^{r' < r} \right]$$

Cross-sectional view



For $0 < r' < r$,
use $\vec{A}_{out}$

For $r < r' < R$
use $\vec{A}_{in}$

$$= \frac{\mu_0 \omega \rho}{3} \sin\theta \, \hat{\phi} \left[\underbrace{\frac{r'^2}{2} r \Big|_r^R}_{\frac{R^2 r - r^3}{2}} + \underbrace{\frac{r'^5}{5r^2} \Big|_0^r}_{\frac{r^3}{5}} \right]$$

$$\underbrace{\frac{R^2 r - r^3}{2} \qquad \frac{r^3}{5}}_{\frac{R^2 r}{2} - \frac{(5-2) r^3}{10}}$$

$$= r\left(\frac{R^2}{2} - \frac{3}{10} r^2\right)$$

$$= \frac{r}{2}\left(R^2 - \frac{3}{5} r^2\right)$$

$$\boxed{\therefore \vec{A}_{in} = \frac{\mu_0 \omega \rho r}{6}\left(R^2 - \frac{3}{5} r^2\right) \sin\theta \, \hat{\phi}}$$

(d) Find $\vec{B}_{in} = \vec{\nabla} \times \vec{A}_{in} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\phi\right)\right]\hat{r}$

$$- \frac{1}{r}\frac{\partial}{\partial r}\left(r A_\phi\right)\hat{\theta}$$

$$\vec{B}_{in} = \frac{\mu_0 \omega \rho}{6}\left\{\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta} r\left(R^2 - \frac{3}{5}r^2\right)\sin^2\theta\right)\hat{r}\right.$$

$$\left. - \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\left(R^2 - \frac{3}{5}r^2\right)\sin\theta\right)\hat{\theta}\right\}$$

$$= \frac{\mu_0 \omega \rho}{6}\left\{\frac{1}{\cancel{r}\cancel{\sin\theta}}\cancel{r}\left(R^2 - \frac{3}{5}r^2\right)2\cancel{\sin\theta}\cos\theta\,\hat{r}\right.$$

$$\left. - \frac{1}{\cancel{r}}\left[2\cancel{r}\left(R^2 - \frac{3}{5}r^2\right)\sin\theta - \frac{6r^{\cancel{3}\,2}}{5}\sin\theta\right]\hat{\theta}\right\}$$

$$= \frac{\mu_0 \omega \rho}{6}\left\{2\left(R^2 - \frac{3}{5}r^2\right)\cos\theta\,\hat{r} - 2\left(R^2 - \frac{3}{5}r^2\right)\sin\theta\,\hat{\theta}\right.$$

$$\left. - \frac{6}{5}r^2\sin\theta\,\hat{\theta}\right\}$$

$$= \frac{\mu_0 \omega \rho}{6} \left\{ 2\left(R^2 - \frac{3}{5} r^2\right) \underbrace{\left(\cos\theta\, \hat{r} - \sin\theta\, \hat{\theta}\right)}_{\hat{z}} - \frac{6}{5} r^2 \sin\theta\, \hat{\theta} \right\}$$

$$\boxed{\therefore\ \vec{B}_{in} = \frac{\mu_0 \omega \rho}{3} \left\{ \left(R^2 - \frac{3}{5} r^2\right) \hat{z} - \frac{3}{5} r^2 \sin\theta\, \hat{\theta} \right\}}$$

A convenient, but slightly unusual expression since in mixes spherical and Cartesian unit vectors. This time, $\vec{B}_{in}$ is not uniform everywhere inside the sphere.

(e) Finally, at the centre of the rotating sphere $r=0$ s.t.

$$\boxed{\vec{B}_{in}(0) = \frac{\mu_0 \omega \rho R^2}{3} \hat{z}}$$

You will obtain this result in Assign. 4.